# Finding roots of $\sin x + x = 0$ and $\sin x - x = 0$

*Anton Shterenlikht*

Mech Eng Dept, The University of Bristol, Bristol BS8 1TR, UK

The above equations have infinite number of roots. I want to find the first $n$ roots. For each root, the minus root is also a root. Also, a conjugate of each root is also a root. I need to use roots with positive real part to ensure rapid decay of the solutions with $x_1$. I will split the complex equations into a system of 2 real equations and then use a standard library to solve it. In the following $j = \sqrt{-1}$.

$x = \operatorname{Re} x + j \operatorname{Im} x$. Let's denote $\operatorname{Re} x = R$, $\operatorname{Im} x = I$. Use the identity $\sin(a+b) = \sin a \cos b + \cos a \sin b$, so that

$$\sin x = \sin(R + jI) = \sin R \cos jI + \cos R \sin jI$$

Now use $\cos a = \frac{1}{2}(\exp ja + \exp(-ja))$, $\sin a = \frac{1}{2j}(\exp ja - \exp(-ja))$:

$$\cos jI = \frac{1}{2}(\exp(jjI) + \exp(-jjI)) \quad ; \quad \sin jI = \frac{1}{2j}(\exp(jjI) - \exp(-jjI))$$

$$\cos jI = \frac{1}{2}(\exp(-I) + \exp I) \quad ; \quad \sin jI = \frac{1}{2j}(\exp(-I) - \exp I)$$

so

$$\sin x = \sin R \frac{1}{2}(\exp(-I) + \exp I) + \cos R \frac{1}{2j}(\exp(-I) - \exp I)$$

Fist $\sin x + x = 0$.

$$\sin x + x = \sin R \frac{1}{2}(\exp(-I) + \exp I) + \cos R \frac{1}{2j}(\exp(-I) - \exp I) + R + jI = 0$$

multiply by $2j$ and split the real and imaginary parts into two real equations:

$$j \sin R(\exp(-I) + \exp I) + \cos R(\exp(-I) - \exp I) + 2jR - 2I = 0$$

$$\sin R(\exp(-I) + \exp I) + 2R = 0$$

$$\cos R(\exp(-I) - \exp I) - 2I = 0$$

Now $\sin x - x = 0$.

$$\sin x - x = \sin R \frac{1}{2}(\exp(-I) + \exp I) + \cos R \frac{1}{2j}(\exp(-I) - \exp I) - R - jI = 0$$

multiply by $2j$ and split the real and imaginary parts into two real equations:

$$j \sin R(\exp(-I) + \exp I) + \cos R(\exp(-I) - \exp I) - 2jR + 2I = 0$$

$$\sin R(\exp(-I) + \exp I) - 2R = 0$$

$$\cos R(\exp(-I) - \exp I) + 2I = 0$$